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Weak Via Strong Stackelberg Problem: New Results

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Abstract. We are concerned with weak Stackelberg problems such as those considered in [19], [23] and [25]. Based on a method due to Molodtsov, we present new results to approximate such problems by sequences of strong Stackelberg problems. Results related to convergence of marginal functions and approximate solutions are given. The case of data perturbations is also considered.

Keywords: Stackelberg problems, Molodtsov's method, approximate solutions, limits of sets, epiconvergence, sequential approximation.

1. Introduction

Let U and V be two topological Hausdorff spaces, X, Y be two non empty subsets respectively of U and V, f_1 be a function from $U \times V$ in $\mathbf{R}$ and f_2 be a function from $U \times V$ in $\mathbf{R} \cup \{+\infty\}$. In this paper we consider the following weak Stackelberg problem

$$S \begin{cases} \operatorname*{Min}_{x \in X} \sup_{y \in M_2(x)} f_1(x,y) \\ \\ \text{where} \quad M_2(x) \quad \text{is the set of optimal solutions to the problem} \\ \\ P(x) : \operatorname*{Min}_{y \in Y} f_2(x,y) \end{cases}$$

$\overline{x} \in X$ solving S will be called a Stackelberg solution to S, any pair $(\overline{x}, \overline{y})$ with $\overline{y} \in M_2(\overline{x})$ and $\overline{x} \in X$ solving S a Stackelberg equilibrium pair and

$$v_1 = \inf_{x \in X} \sup_{y \in M_2(x)} f_1(x,y)$$

the value of S.

To motivate such a study we refer to some previous works ([5], [15], [19], [25], [30], [34]).

The problem S may have no solutions even for nice functions f_1 and f_2, so the following regularized problem $S(\varepsilon)$ has been considered for $\varepsilon > 0$.

$$S(\varepsilon)\begin{cases} \operatorname*{Min}_{x\in X} \sup_{y\in M_2(x,\varepsilon)} f_1(x,y) \\ \text{where } M_2(x,\varepsilon) \text{ is the set of } \varepsilon\text{-solutions to} \\ P(x) : \operatorname*{Min}_{y\in Y} f_2(x,y) \end{cases}$$

Let us point out that sufficient conditions of minimal character ensuring well-posedness and existence and stability of the solutions to the regularized problem under data perturbations have been given in [23], [25], [26], [27], [28], [34] ... In our opinion these theoretical results get an insight into the inherent difficulties of the problem and can explain the lack of non heuristic numerical methods in the continuous case.

Nevertheless there is an other kind of Stackelberg problem, namely the strong Stackelberg one, which appears to be best handled:

$$\begin{cases} \operatorname*{Min}_{x\in X} \inf_{y\in M_2(x)} f_1(x,y) \\ \text{where } M_2(x) \text{ is the set of optimal solutions to} \\ P(x) \quad \begin{cases} \operatorname*{Min}_{y\in Y} f_2(x,y) \end{cases} \end{cases}$$

In fact, for such a problem under inequality constraints, in addition to existence and stability results ([24]) there are different papers on necessary and sufficient conditions [8], [11], [36], [39], [40] and numerical methods ([2], [4], [6], [10], [12], [13], [14], [17], [35], [37], [38] ...). But the method described by Molodtsov (in [33]) which approaches the weak problem S by a sequence of strong Stackelberg problems could be a first step towards the numerical resolution of the problem S. So it appeared useful to us to go further on the Molodtsov results. First results in this direction have been given in [29] and [1].

In this paper, we first recall the Molodtsov method and its convergence results on the values of the strong approximate problem, obtained for continuous functions.

Then in section 3 we present complementary results on a regularized version of the problem (S) already considered in [15] and [26], [27], [28]. These results allow to present in section 4 more general properties on Molodtsov approximation under assumptions of minimal character as well on the values as on the solutions.

Then, in order to open a way for the use of numerical approximations (such that discretizations and penalizations), perturbations on the data of the problem (S) will be considered, in section 5, for what concerns Molodtsov values. In section 6 convergences of Molodtsov marginal functions and approximate solutions under perturbations will be given.

Let us note there is a gap between the results obtained without perturbations and under perturbations. For example, as shown in section 6, there is a class of data for which the convergence of the solutions to the approximate unperturbed problems to an exact lower Stackelberg equilibrium is guaranteed but when perturbations are involved the convergence is obtained only to a lower Stackelberg equilibrium pair.

2. Molodtsov results

First of all we recall the method introduced by Molodtsov in [33]. For $\beta \geq 0$ let:

$$g(x,y,\beta) = f_2(x,y) - \beta f_1(x,y) \quad \text{for any} \quad x \in X \quad \text{and} \quad y \in Y .$$

For $\beta \geq 0$ and $\gamma \geq 0$ the following strong Stackelberg problem can be defined:

$$Q^{\beta,\gamma} \begin{cases} \operatorname*{Min}_{x \in X} \inf_{y \in M(x,\beta,\gamma)} f_1(x,y) \\ \text{where } \; M(x,\beta,\gamma) \; \text{ is the set of } \gamma\text{-solutions} \\ \text{to the parametrized problem:} \\ R^\beta(x) : \operatorname*{Min}_{y \in Y} g(x,y,\beta) \end{cases}$$

In the sequel, we shall let $v_2(x) = \inf_{y \in Y} f_2(x,y)$.
The following two propositions are nothing but an adaptation of the results in [33].

PROPOSITION 1 *Let $\beta \geq 0$, $\gamma \geq 0$ and assume:*

(2.1) $v_2(x)$ *is a real finite number for any* $x \in X$,
(2.2) *there exists* $c > 0$ *such that* $\mid f_1(x,y) \mid \leq c$ *for any* $x \in X$ *and any* $y \in Y$;

then $M(x,\beta,\gamma)$ *is a non empty set for any* $x \in X$ *and* $M(x,\beta,\gamma) \subseteq M_2(x,\gamma+2\beta c)$.

Proof: $g(x,y,\beta) \geq v_2(x) - \beta c > -\infty$ for any $x \in X$ and $y \in Y$ then $M(x,\beta,\gamma) = \{y \in Y : g(x,y,\beta) \leq \inf_{z \in Y} g(x,z,\beta) + \gamma\}$ is non empty.

Moreover $f_2(x,y) - \beta c \leq g(x,y,\beta)$ for any $y \in Y$ and, for $y \in M(x,\beta,\gamma)$, $g(x,y,\beta) \leq f_2(x,z) - \beta f_1(x,z) + \gamma$ for any $z \in Y$. Then we deduce:

$$f_2(x,y) - \beta c \leq v_2(x) + \beta c + \gamma \quad \text{for any} \quad y \in M(x,\beta,\gamma)$$

and the result follows.

∎

Define, for $\varepsilon \geq 0$

$$(2.3) \quad w_1(x,\varepsilon) = \sup_{y \in M_2(x,\varepsilon)} f_1(x,y) \qquad v_1(\varepsilon) = \inf_{x \in X} w_1(x,\varepsilon)$$

$$\text{and} \quad w_1(x,0) = w_1(x) \qquad v_1(0) = v_1$$

we have:

PROPOSITION 2 *Let $\varepsilon \geq 0$, $\beta > 0$, $\gamma \geq 0$ and assume that assumptions (2.1), (2.2) are satisfied, then:*

$$(2.4) \quad f_1(x,y) \leq w_1(x,\gamma + 2\beta c) \quad \text{for any} \quad y \in M(x,\beta,\gamma)$$

$$(2.5) \quad w_1(x,\varepsilon) - \frac{\varepsilon+\gamma}{\beta} \leq f_1(x,y) + \frac{1}{\beta}(v_2(x) - f_2(x,y)) \leq f_1(x,y)$$

for any $x \in X$ and any $y \in M(x,\beta,\gamma)$.

Proof: (2.4) is a direct consequence of Proposition 1. Moreover, let $y \in M(x,\beta,\gamma)$ then:

$$\begin{aligned} g(x,y,\beta) &\leq \inf_{z \in Y} [f_2(x,z) - \beta f_1(x,z)] + \gamma \\ &\leq \inf_{z \in M_2(x,\varepsilon)} [f_2(x,z) - \beta f_1(x,z)] + \gamma \\ &\leq \inf_{z \in M_2(x,\varepsilon)} [v_2(x) + \varepsilon - \beta f_1(x,z)] + \gamma \\ &\leq v_2(x) + \varepsilon - \beta w_1(x,\varepsilon) + \gamma \ . \end{aligned}$$

Hence we get:

$$\beta w_1(x,\varepsilon) - (\gamma+\varepsilon) \leq v_2(x) - f_2(x,y) + \beta f_1(x,y) \leq \beta f_1(x,y)$$

for any $x \in X$, any $y \in M(x,\beta,\gamma)$.

■

Remark. If $M_2(x) \neq \emptyset$ then (2.5) holds for $\varepsilon = 0$

$$w_1(x) - \frac{\gamma}{\beta} \leq f_1(x,y) + \frac{1}{\beta}[v_2(x) - f_2(x,y)] \leq f_1(x,y) \ .$$

If $M_2(x) = \emptyset$ then $w_1(x) = -\infty$ and the previous inequalities are trivially satisfied.

Now let us introduce the marginal function of the first level problem in the problem $Q^{\beta,\gamma}$ for $\beta > 0$, $\gamma \geq 0$

$$\begin{cases} m_1(x,\beta,\gamma) = \inf_{y \in M(x,\beta,\gamma)} f_1(x,y) \\ t_1(\beta,\gamma) = \inf_{x \in X} m_1(x,\beta,\gamma) \ . \end{cases} \qquad (2.6)$$

The following proposition is obvious.

PROPOSITION 3 *Let* $\varepsilon \geq 0$, $\beta > 0$, $\gamma \geq 0$.
Under the assumptions (2.1) and (2.2) we get:

$$\text{(2.7)} \quad w_1(x,\varepsilon) - \frac{\varepsilon+\gamma}{\beta} \leq m_1(x,\beta,\gamma) \leq \sup_{y \in M(x,\beta,\gamma)} f_1(x,y) \leq w_1(x,\gamma+2\beta c)$$

$$\text{(2.8)} \quad v_1(\varepsilon) - \frac{\varepsilon+\gamma}{\beta} \leq t_1(\beta,\gamma) \leq \inf_{x \in X} \sup_{y \in M(x,\beta,\gamma)} f_1(x,y) \ldots \leq v_1(\gamma+2\beta c) \ .$$

Remark. In [33, Theorem 1] it is proved that if X and Y are metric compact spaces with f_1 and f_2 continuous on $X \times Y$, then:

$$\lim_{n\to\infty} t_1(\beta_n,\gamma_n) = v_1 \quad \text{if} \quad \beta_n \to 0^+, \quad \gamma_n \to 0^+ \ \text{with} \ \frac{\gamma_n}{\beta_n} \to 0^+ \ .$$

We shall prove a similar result under weaker assumptions by using some properties of the regularized weak Stackelberg problem $S(\varepsilon)$.

3. On the regularized weak Stackelberg problem $S(\varepsilon)$

First of all we recall some useful properties of $v_1(\varepsilon)$, the value of the problem $S(\varepsilon)$ considered in the introduction.

PROPOSITION 4 *([28], Proposition 2.4)*
Suppose that Y is sequentially compact and

(3.1) the function f_2 is sequentially lower semicontinuous on $X \times Y$.

(3.2) For any $(x,y) \in X \times Y$ and any sequence $(x_n)_n$ converging to x there exists a sequence $(y_n)_n$ such that:

$$\limsup_{n\to\infty} f_2(x_n,y_n) \leq f_2(x,y) \ .$$

(3.3) For any $x \in X$ there exists a sequence $(x_n)_n$ converging to x such that for any $y \in Y$ and any sequence $(y_n)_n$ converging to y we have:

$$\limsup_{n\to\infty} f_1(x_n,y_n) \leq f_1(x,y)$$

then $\lim_{n\to\infty} v_1(\varepsilon_n) = v_1$ *for any* $\varepsilon_n \to 0^+$.

Proof: Recalled for the sake of convenience of the reader. First of all, we notice that, for any $x \in X$ and any $\varepsilon > 0$,

$$M_2(x) \subseteq M_2(x, \varepsilon) .$$

We can deduce that $w_1(x) \leq w_1(x, \varepsilon)$ and $v_1 \leq v_1(\varepsilon)$ for any $\varepsilon \geq 0$. So we get:

$$v_1 \leq \liminf_{n \to \infty} v_1(\varepsilon_n) \quad \text{for any} \quad \varepsilon_n \to 0^+ .$$

Now, in order to prove that $\limsup_{n \to \infty} v_1(\varepsilon_n) \leq v_1$ for any sequence $\varepsilon_n \to 0^+$ we have to prove the following property (see [20], Proposition 3.1.1): for any $x \in X$ and for any $\varepsilon_n \to 0^+$ there exists a sequence $(x_n)_n$ in X such that:

$$\limsup_{n \to \infty} w_1(x_n, \varepsilon_n) \leq w_1(x) .$$

Let $x \in X$ and $(x_n)_n$ be a sequence as defined in the assumption (3.3). Assume the previous inequality is false, that is there exists a real a such that:

$$w_1(x) < a < \limsup_{n \to \infty} w_1(x_n, \varepsilon_n) .$$

Then, there exists a subsequence $(w_1(x_{n_k}, \varepsilon_{n_k}))_k$ verifying $w_1(x_{n_k}, \varepsilon_{n_k}) > a$ for any $k \in \mathbf{N}$. Therefore there exists a sequence $(y_n)_n$ such that $y_{n_k} \in M_2(x_{n_k}, \varepsilon_{n_k})$ and $f_1(x_{n_k}, y_{n_k}) > a$ for any $k \in \mathbf{N}$. From the sequential compactness of Y, there exists a subsequence $(y_{n_{k_j}})_j$, converging to $\overline{y}$ such that:

$$f_2(x_{n_{k_j}}, y_{n_{k_j}}) \leq v_2(x_{n_{k_j}}) + \varepsilon_{n_{k_j}} .$$

From (3.1) we get:

$$f_2(x, \overline{y}) \leq \limsup_{j \to \infty} v_2(x_{n_{k_j}}) \leq \limsup_{n \to \infty} v_2(x_n) .$$

But the assumption (3.2) is equivalent (see again Proposition 3.1 in [20]) to the following:

$$\limsup_{n \to \infty} v_2(x_n) \leq v_2(x) \quad \text{then} \quad \overline{y} \in M_2(x) .$$

Therefore, by using (3.3), we have:

$$a \leq \limsup_{j \to \infty} f_1(x_{n_{k_j}}, y_{n_{k_j}}) \leq f_1(x, \overline{y}) \leq w_1(x)$$

which contradicts $w_1(x) < a$. We have just proved that $\lim_{n \to \infty} v_1(\varepsilon_n) = v_1$ and it is easy to conclude that $\lim_{\varepsilon \to 0^+} v_1(\varepsilon) = v_1$.

∎

Remark. Let us recall that in [33] this result is obtained under continuity of the functions f_1 and f_2.

Now, let us recall a pointwise convergence result on $w_1(\cdot, \varepsilon_n)$ to w_1, for ε_n decreasing to zero, which is merely an adaptation of well known results (see, for example, [16]).

PROPOSITION 5 *Let Y be a sequentially compact space and assume the following are satisfied:*

(3.4) the function $y \to f_2(x,y)$ is sequentially lower semicontinuous on Y for any $x \in X$;

(3.5) the function $y \to f_1(x,y)$ is sequentially semicontinuous on Y for any $x \in X$;

then

i) $\lim_{n\to\infty} w_1(x,\varepsilon_n) = w_1(x)$ *for any $x \in X$ and any ε_n decreasing to zero;*

ii) $\lim_{n\to\infty} v_1(\varepsilon_n) = v_1$ *for any ε_n decreasing to zero.*

Proof: Prove i). For sake of convenience, we shall give a direct proof.
First of all, from condition (3.4) and the sequentially compactness of the space Y we notice that $M_2(x) \neq \emptyset$ and $v_2(x)$ is a finite number for any $x \in X$.
For $\varepsilon > 0$ and fixed $x \in X$, the problem $\{ \operatorname*{Sup}_{y \in M_2(x,\varepsilon)} f_1(x,y)$ can be seen as a perturbed maximization problem with right-hand side perturbations due to the parameter ε. Now, let $\varepsilon_n \searrow 0^+$. It is easy to see, from (3.4), that for any fixed $x \in X$, the multifunction $M_2(x,\cdot)$ is sequentially closed at 0 (with $M_2(x,0) = M_2(x)$). Furthermore, for any $n \in N$, there exists $y_n \in M_2(x,\varepsilon_n)$ such that:

$$f_1(x,y_n) = w_1(x,\varepsilon_n) \ .$$

From sequential compactness of Y there exists a subsequence $(y_{n_k})_k$ converging to y_0 with $y_0 \in M_2(x)$ and from (3.5) we get:

$$\limsup_{k\to\infty} w_1(x,\varepsilon_{n_k}) = \limsup_{k\to\infty} f_1(x,y_{n_k}) \le f_1(x,y) \le w_1(x) \ .$$

Now, since we also have $w_1(x) \le w_1(x,\varepsilon_n)$ for any n, it is easy to conclude that:

$$\lim_{k\to\infty} w_1(x,\varepsilon_{n_k}) = w_1(x) \ .$$

Finally, by a classical argument, we deduce that:

$$\lim_{n\to\infty} w_1(x,\varepsilon_n) = w_1(x)$$

for any $x \in X$ and any $\varepsilon_n \searrow 0^+$.

Prove ii). From i) and proposition 3.1.1 in [20]

$$\limsup_{n\to\infty} v_1(\varepsilon_n) \le v_1$$

and the result follows.

■

Remark. ii) of proposition 5 gives an alternative result for the convergence of the regularized value $v_1(\varepsilon)$. In fact, assumptions (3.1) and (3.2) are stronger than assumption (3.4) but assumption (3.3) is weaker than assumption (3.5).

In the following, for any sequence of sets A_n, $n \in \mathbf{N}$, in a space Y,

$$\operatorname*{Limsup}_n A_n = \{y \in Y : \text{ there exists a sequence } (y_k) \text{ converging to } y \text{ such that } y_k \in A_{n_k} \text{ for a selection of integers } (n_k)\}$$

PROPOSITION 6 *Under assumptions (3.4) and (3.5) we have: for any $\varepsilon_n \searrow 0^+$*

$$\operatorname*{Limsup}_n M_1(\varepsilon_n) \subseteq \overline{M}_1 \ ,$$

where $M_1(\varepsilon)$ denotes the set of the solutions to $S(\varepsilon)$ for any $\varepsilon > 0$ and $\overline{M}_1$ is the set of the solutions to the lower semicontinuous regularized problem $\overline{S}$ defined by

$$\overline{S}\begin{cases}\operatorname*{Min}_{x\in X} \overline{\sup_{y\in M_2(x)} f_1(x,y)}\end{cases}$$

where $\overline{\sup_{y\in M_2(x)} f_1(x,y)}$ is the lower semicontinuous regularized function of $\sup_{y\in M_2(x)} f_1(x,y)$ that is: if $\overline{x}_n$ is a solution to $S(\varepsilon_n)$ and $(\overline{x}_{n_k})_k$ is convergent to $\overline{x}$ for a selection of integers (n_k) then we have: $\overline{x}$ is a solution to $\overline{S}$.

Proof: Let $g_n(x) = w_1(x,\varepsilon_n) = \sup_{y\in M_2(x,\varepsilon_n)} f_1(x,y)$ and $g(x) = \overline{w}(x)$ for any $x \in X$.

It is sufficient to verify that g_n is epiconvergent (or Γ^--convergent) to g, that is ([9], [7], [3])

- for any $x \in X$ and any $(x_n)_n$ converging to x we have: $\liminf_{n\to\infty} g_n(x_n) \ge g(x)$
- for any $x \in X$ there exists $(\overline{x}_n)_n$ converging to x such that: $\limsup_{n\to\infty} g_n(\overline{x}_n) \le g(x)$,

and to apply, for example, proposition 2.3.1 in [21].

Now let ε_n be a sequence decreasing to zero. Since $w_1(x,\varepsilon_n) \ge w_1(x,\varepsilon_{n+1})$ for any $x \in X$ and any $n \in \mathbf{N}$, we know from [3] that: $w_1(\cdot,\varepsilon_n)$ is epiconvergent to

$\overline{\inf_n w_1(\cdot,\varepsilon_n)}$ and $w_1(\cdot,\varepsilon_n)$ is epiconvergent to $\overline{w}_1$

■

In the next proposition we recall conditions ensuring that $M_1(\varepsilon)$ is non empty.

PROPOSITION 7 *Let X and Y be two sequentially compact spaces and Y be a convex space. Assume that (3.1) and the following are satisfied:*

(3.2e) For any $(x,y) \in X \times Y$ and any sequence $(x_n)_n$ converging to x there exists a sequence $(y_n)_n$ converging to x such that:

$$\limsup_{n\to\infty} f_2(x_n,y_n) \leq f_2(x,y) .$$

(3.6) The function $y \to f_2(x,y)$ is strictly quasiconvex ([32]) on Y for any $x \in X$.

(3.7) The function f_1 is sequentially lower semicontinuous on $X \times Y$

then, for any $\varepsilon > 0$, there exists at least a solution x_ε to the problem $S(\varepsilon)$.

Proof: Is a consequence of Proposition 2.2 in [28]. Again, for sake of convenience, let us recall the proof.

Proof of the Proposition 7
In the following $\overline{A}^s$ denotes the sequential closure of A (that is $y \in \overline{A}^s$ if and only if there exists a sequence $(y_n)_n$ in A converging to y) and $\operatorname{Liminf}_n A_n = \{y \in Y /$ there exists a sequence $(y_n)_n$ converging to y such that $y_n \in A_n$ for any $n \in \mathbf{N}\}$.

In order to prove Proposition 7 it is sufficient to prove that, for any $\varepsilon > 0$ and any sequence $(x_n)_n$ converging to x, we have:

$$(3.8) \quad M_2(x,\varepsilon) \subseteq \overline{\operatorname{Liminf}_n M_2(x_n,\varepsilon)}^s$$

and, taking into account the condition (3.7), to apply Proposition 2.3.1 in [22]. In fact let $\widetilde{M_2}(x,\varepsilon) = \{y \in Y / f_2(x,y) < v_2(x) + \varepsilon\}$ the set of the strict ε-solutions to the problem $P(x)$. Under the assumption (3.6), let us prove that

$$(3.9) \quad M_2(x,\varepsilon) \subseteq \overline{\widetilde{M_2}(x,\varepsilon)}^s .$$

Let $y \in M_2(x,\varepsilon)$ such that $y \neq \widetilde{M_2}(x,\varepsilon)$, $\tilde{y}_0 \in \widetilde{M_2}(x,\varepsilon)$ and $\tilde{y}_n = \dfrac{1}{n}\tilde{y}_0 + \left(1 - \dfrac{1}{n}\right) y$. Then $(\tilde{y}_n)_n$ is convergent to $\tilde{y}$,

$$f_2(x,\tilde{y}_n) < \max(f_2(x,\tilde{y}_0,\ f_2(x,y)) \leq v_2(x) + \varepsilon$$

and $\tilde{y}_n \in \widetilde{M_2}(x,\varepsilon)$.

Therefore $y \in \overline{\widetilde{M}_2(x,\varepsilon)}^s$ and (3.9) is proved. But for any sequence $(x_n)_n$ converging to x it results:

$$(3.10)\widetilde{M}_2(x,\varepsilon) \subseteq \operatorname*{Liminf}_n \widetilde{M}_2(x_n,\varepsilon) .$$

In fact, let $y \in \widetilde{M}_2(x,\varepsilon)$. From (3.2e) there exists a sequence $(y_n)_n$ converging to y such that $\limsup_{n\to\infty} f_2(x_n,y_n) \le f_2(x,y)$ and, from (3.1) and (3.2e) we have (see, for example, Proposition 3.1.1 and 4.1.1 in [20]):

$$(3.11) \quad \lim_{n\to\infty} v_2(x_n) = v_2(x) .$$

So, for n sufficiently large, $f_2(x_n,y_n) < v_2(x_n) + \varepsilon$ that is $y_n \in \widetilde{M}_2(x_n,\varepsilon)$ and (3.10) is satisfied.
Then we have:

$$\begin{aligned} M_2(x,\varepsilon) &\subseteq \overline{\widetilde{M}_2(x,\varepsilon)} \subset \overline{\operatorname*{Liminf}_n \widetilde{M}_2(x_n,\varepsilon)}^s \\ &\subseteq \overline{\operatorname*{Liminf}_n M_2(x_n,\varepsilon)}^s \end{aligned}$$

■

Remark. Under the assumptions of Proposition 5, if w_1 is lower semicontinuous in X we have:

$$\operatorname*{Limsup}_{n\to\infty} M_1(\varepsilon_n) \subseteq M_1 \qquad \text{the set of the solutions to } S$$

for any $\varepsilon_n \searrow 0^+$.
Moreover, if there exists a unique solution $\overline{x}$ to S then:

$$\operatorname*{Limsup}_n M_1(\varepsilon_n) = \{\overline{x}\} \quad \text{for any} \quad \varepsilon_n \searrow 0^+ .$$

4. New result on Molodtsov regularization

First, by using Proposition 4 or Proposition 5, let us improve Theorem 1 in [33].

PROPOSITION 8 *Assume conditions (2.1), (2.2) and (3.1) to (3.3) (or (3.4) and (3.5)) are satisfied, we have:*

$$\lim_{n\to\infty} t_1(\beta_n,\gamma_n) = v_1 \quad \textit{for any} \quad \beta_n \searrow 0^+ , \quad \gamma_n \searrow 0^+ \quad \textit{such that } \frac{\gamma_n}{\beta_n} \to 0^+$$

Proof: Let $\beta_n \searrow 0^+$, $\gamma_n \searrow 0^+$ such that $\frac{\gamma_n}{\beta_n} \to 0^+$. From (2.8) with $\varepsilon = 0$

$$v_1 - \frac{\gamma_n}{\beta_n} \le t_1(\beta_n,\gamma_n) \le v_1(\gamma_n + 2\beta_n c) .$$

We get the result by using Proposition 4 (or Proposition 5). ■

Remark. In the previous result X is not necessarily sequentially compact. Moreover (3.2) and (3.3) are even weaker than the following conditions:

(4.1) the function $y \to f_2(x, y)$ is sequentially upper semicontinuous for any $x \in X$;

(4.2) the function $y \in f_1(x, y)$ is upper semicontinuous for any $x \in X$.

Now, we are interested in existence of solutions to the problem $Q^{\beta,\gamma}$ and their connections with solutions to S.

PROPOSITION 9 *Let X and Y be two sequentially compact spaces. Assume assumptions (3.1), (3.2e) and the following are satisfied:*

(4.3) f_1 is sequentially continuous on $X \times Y$

then there exists at least a solution to the problem $Q^{\beta,\gamma}$ for any $\beta \geq 0$ and $\gamma \geq 0$.

Proof: It is sufficient to have the function f_1 sequentially lower semicontinuous on $X \times Y$ and the multifunction $M(\cdot, \beta, \gamma)$ sequentially closed graph (see [20] for example) in order to obtain the marginal function $m_1(\cdot, \beta, \gamma)$, as defined by (2.6), sequentially lower semicontinuous (see, for example, Proposition 4.2.1 in [20]). But, under the assumptions, the function $g = f_2 - \beta f_1$ satisfies the following conditions:

(i) g is sequentially lower semicontinuous in $X \times Y$

(ii) for any $(x, y) \in X \times Y$ and any sequence $(x_n)_n$ converging to x there exists a sequence $(y_n)_n$ such that

$$\limsup_{n\to\infty} g(x_n, y_n, \beta) \leq g(x, y, \beta)$$

which ensures that $M(\cdot, \beta, \gamma)$ is sequentially closed graph on X.

■

Remark. If $\beta = 0$ we get the regularized strong Stackelberg problem ([24]).

In order to study the convergence of optimal solutions to Q^{β_n,γ_n}, for $\beta_n \searrow 0^+$ and $\gamma_n \searrow 0^+$, let us give pointwise convergence and epiconvergence results for the marginal function m_1 of the upper level in the problem Q^{β_n,γ_n}, as defined by (2.6).

PROPOSITION 10 *Let Y be a sequentially compact space and assume that (2.1), (2.2), (3.4) and (3.5) are satisfied. Then, for any $\beta_n \searrow 0^+$ and $\gamma_n \searrow 0^+$ such that $\dfrac{\gamma_n}{\beta_n} \to 0^+$ we have:*

(4.4) $$m_1(x, \beta_n, \gamma_n) \to w_1(x) \quad \textit{for any} \quad x \in X\ .$$

Proof: Obvious from (2.7) with $\varepsilon = 0$ and i) of Proposition 5. ■

Now, let us denote:

(4.5) $m_{1,n}(x) = m_1(x, \beta_n, \gamma_n)$ for any $x \in X$ and any $n \in \mathbf{N}$

(4.6) $N_n(\alpha) = \{x \in X / m_{1,n}(x) \leq t_1(\beta_n, \gamma_n) + \alpha\}$ for $\alpha \geq 0$.

Recall $\overline{M}_1 = \{x \in X / \overline{w}_1(x) = v_1\}$, where

$$\overline{w_1}(x) = \liminf_{z \to x} w_1(z) \ .$$

PROPOSITION 11 *Let Y be a sequentially compact space and assume conditions (2.1), (2.2), (3.4) and (3.5) are satisfied. Let (β_n) and $(\gamma_n)_n$ be two sequences of real positive numbers decreasing to zero such that $\lim_{n\to\infty} \frac{\gamma_n}{\beta_n} = 0$, then: $m_{1,n}$ is epiconvergent to w_1.*

Proof: From (2.7)

$$\overline{w}_1(x) - \frac{\gamma_n}{\beta_n} \leq w_1(x) - \frac{\gamma_n}{\beta_n} \leq m_{1,n}(x)$$

then we get, for any x and any sequence $(x_n)_n$ converging to x:

$$\overline{w}_1(x) \leq \liminf_{n\to\infty} \overline{w}_1(x_n) \leq \liminf_{n\to\infty} m_{1,n}(x_n) \ .$$

Moreover, in the proof of Proposition 6, we proved that $w_1(\cdot, \varepsilon_n)$ is epiconvergent to $\overline{w}_1$ for any ε_n decreasing to zero.

So, for any $\overline{x} \in X$, there exists $(\overline{x}_n)_n$ converging to x such that:

$$\limsup_{n\to\infty} w_1(\overline{x}_n, \gamma_n + 2\beta_n c) \leq \overline{w}_1(x) \ .$$

Now $m_{1,n}(z) \leq w_1(z, \gamma_n + 2\beta_n c)$ for any $z \in X$ and $\limsup_{n\to\infty} m_{1,n}(\overline{x}_n) \leq \overline{w}_1(x)$. ■

COROLLARY 1 *Under the assumptions of Proposition 11 we have:*

$$\operatorname*{Limsup}_n N_n(\alpha_n) \subseteq \overline{M}_1 \qquad \textit{for any } \alpha_n \searrow 0^+ \ .$$

PROPOSITION 12 *Let $\varepsilon_n \searrow 0^+$, $\beta_n \searrow 0^+$, $\gamma_n \searrow 0^+$, $\alpha_n \searrow 0^+$ such that $\frac{\varepsilon_n}{\beta_n} \to 0^+$ and $\frac{\gamma_n}{\beta_n} \to 0^+$. Let Y be sequentially compact. Under assumptions (2.1), (2.2), (3.1), (3.2), (3.3), if $x_n \in N_n(\alpha_n)$, $y_n \in M_2(x_n, \varepsilon_n)$ and $(x_n, y_n)_n$ is convergent to (x^*, y^*) then (x^*, y^*) is a lower Stackelberg equilibrium for S that is ([25]):*

$$f_1(x^*, y^*) \le v_1 \quad \textit{and} \quad y^* \in M_2(x^*) \ .$$

Proof: If $x_n \in N_n(\alpha_n)$ and $y_n \in M_2(x_n, \varepsilon_n)$ we have:

$$m_{1,n}(x_n) \le t_1(\beta_n, \gamma_n) + \alpha_n$$

and

$$f_1(x_n, y_n) \le w_1(x_n, \varepsilon_n) = \sup_{y \in M_2(x_n, \varepsilon_n)} f_1(x_n, y)$$

but, from (2.7),

$$w_1(x_n, \varepsilon_n) \le m_1(x_n, \beta_n, \gamma_n) + \frac{\varepsilon_n + \gamma_n}{\beta_n}$$

then $f_1(x_n, y_n) \le t_1(\beta_n, \gamma_n) + \alpha_n + \dfrac{\varepsilon_n + \gamma_n}{\beta_n}$ and, by using Proposition 8 and the sequential lower semicontinuity of f_1, we obtain:

$$f_1(x^*, y^*) \le v_1 \ .$$

Finally, from conditions (3.1) and (3.2), we can deduce that $y^* \in M_2(x^*)$

■

In the next proposition $M(x, \beta, 0)$ is supposed to be a singleton for any $x \in X$. This allows to obtain as accumulation point an exact lower Stackelberg equilibrium pair ([1]) that is a point $(\overline{x}, \overline{y})$ such that:

$$\overline{y} \in M_2(\overline{x}) \quad \text{and} \quad f_1(\overline{x}, \overline{y}) = v_1 \ .$$

Proposition 13 *Suppose (2.1), (2.2), (3.1), (3.2), f_1 continuous on $X \times Y$ and*

$$(4.7) \quad M(x, \beta, 0) = \{y(x, \beta)\} \quad \textit{for any} \quad x \in X, \quad \beta \ge 0 \ .$$

Let $\alpha_n \searrow 0^+$, $\beta_n \searrow 0^+$, $x_n \in N_n(\alpha_n)$ and $y_n = y(x_n, \beta_n)$. If $(x_n, y_n)_n$ is convergent to (x^*, y^*) then (x^*, y^*) is an exact lower Stackelberg equilibrium pair for S.

Proof: From Proposition 12 it is sufficient to prove that $f_1(x^*, y^*) \ge v_1$.
But $y_n(x_n, \beta_n) \in M_2(x_n, 2\beta_n c)$ (Proposition 1) then

$$t_1(\beta_n) = \inf_{x \in X} f_1(x, y(x, \beta_n)) \le f_1(x_n, y_n)$$

and, from Proposition 8, we get the result.

■

Remark. For example, assumption (4.7) is satisfied if the function $y \to f_2(x, y)$ is convex and the function $y \to f_1(x, y)$ is strictly concave.

5. Molodtsov value under perturbations

We are interested to consider perturbations on the data of the problem S in order to open a way for the use of numerical approximation methods.

So let us consider two sequences $(f_{i,n})$ for $i = 1, 2$ of extended real valued functions on $X \times Y$, the perturbed regularized weak Stackelberg problem:

$$S_n(\varepsilon) \quad \begin{cases} \operatorname*{Min}_{x \in X} \sup_{y \in M_{2,n}(x,\varepsilon)} f_{1,n}(x,y) \\ \text{where } \ M_{2,n}(x,\varepsilon) \ \text{ is the set of } \varepsilon - \text{solutions to the problem:} \\ P_n(x) \ \begin{cases} \operatorname*{Min}_{y \in Y} f_{2,n}(x,y) \end{cases} \end{cases}$$

and the perturbed Molodtsov regularized Stackelberg problem:

$$(Q_n^{\beta,\gamma}) \quad \begin{cases} \operatorname{Min}_{x \in X} \inf_{y \in M_{n(x,\beta,\gamma)}} f_{1,n}(x,y) \\ \text{where } \ M_n(x,\beta,\gamma) \ \text{ is the set of } \gamma\text{-solutions to the problem:} \\ R_n^{\beta}(x) \ \begin{cases} \operatorname*{Min}_{y \in Y} f_{2,n}(x,y) - \beta f_{1,n}(x,y) \ . \end{cases} \end{cases}$$

Let us denote

$$v_{2,n}(x) = \inf_{y \in Y} f_{2,n}(x,y) \ ,$$
$$v_{1,n}(\varepsilon) = \inf_{x \in X} \sup_{y \in M_{2,n}(x,\varepsilon)} f_{1,n}(x,y) \ ,$$
$$t_{1,n}(\beta,\gamma) = \inf_{x \in X} \inf_{y \in M_n(x,\beta,\gamma)} f_{1,n}(x,y) \ ;$$

$t_{1,n}(\beta,\gamma)$ will be called the perturbed Molodtsov value.

First let give some convergence results for $t_{1,n}(\beta,\gamma_n)$ when β is a fixed positive number and $\gamma_n \searrow 0^+$.

PROPOSITION 14 *Under the following assumptions: X is sequentially compact and*

(5.1) $v_1, v_{2,n}(x)$ and $v_2(x)$ are real finite numbers for any $x \in X$ and any $n \in \mathbf{N}$;

(5.2) the sequence $f_{1,n}$ is equibounded on $X \times Y$ that is to say: there exists $c > 0$ such that $|f_{1,n}(x,y)| \leq c$ for $n \in \mathbf{N}$ and for any $(x,y) \in X \times Y$;

(5.3) for any $(x,y) \in X \times Y$, for any sequence $(x_n, y_n)_n$ converging to (x,y) we have

$$\liminf_{n \to \infty} f_{2,n}(x_n, y_n) \geq f_2(x,y) \ ;$$

(5.4) *for any* $(x, y) \in X \times Y$ *and any sequence* $(x_n)_n$ *converging to* x *in* X *there exists a sequence* $(y_n)_n$ *in* Y *such that*

$$\limsup_{n\to\infty} f_{2,n}(x_n, y_n) \leq f_2(x, y) \ ;$$

(5.5) *for any* $(x, y) \in X \times Y$ *and any sequence* $(x_n, y_n)_n$ *converging to* (x, y) *we have*

$$\liminf_{n\to\infty} f_{1,n}(x_n, y_n) \geq f_1(x, y) \ ;$$

(5.6) *for any* $x \in X$ *there exists a sequence* $(\overline{x}_n)_n$ *converging to* x *such that for any* $y \in Y$ *and any sequence* $(y_n)_n$ *converging to* y*:*

$$\limsup_{n\to\infty} f_{1,n}(\overline{x}_n, y_n) \leq f_1(x, y) \ ,$$

then

(5.7) $v_1 - \dfrac{\varepsilon}{\beta} \leq \limsup\limits_{n\to\infty} t_{1,n}(\beta, \gamma_n) \leq v_1(2\beta c)$ *for any* $\varepsilon > 0$ *and* $\beta > 0$, *and* $\limsup\limits_{n\to\infty} t_{1,n}(\beta_n, \gamma_n) \leq v_1$.

Proof: Let us denote

$$(5.8) \qquad m_{1,n}(x, \beta, \gamma) = \inf_{y \in M_n(x,\beta,\gamma)} f_{1,n}(x, y)$$

we shall call the perturbed Molodtsov marginal function. As in Proposition 3 we can prove, under assumptions (5.1) and (5.2) that, for $\varepsilon \geq 0$, $\gamma \geq 0$ and $\beta > 0$:

$$(5.9) \qquad w_{1,n}(x, \varepsilon) - \frac{\varepsilon + \gamma}{\beta} \leq m_{1,n}(x, \beta, \gamma) \leq w_{1,n}(x, \gamma + 2\beta c)$$

$$(5.10) \qquad v_{1,n}(\varepsilon) - \frac{\varepsilon + \gamma}{\beta} \leq t_{1,n}(\beta, \gamma) \leq v_{1,n}(\gamma + 2\beta c) \ .$$

Let γ_n converging to zero. From Proposition 5.3 and Remark 5.3 in [26], under the assumptions (5.1) and (5.3) to (5.5) we have:

$$v_1 \leq \limsup_{n\to\infty} v_{1,n}(\varepsilon) \quad \text{for any} \quad \varepsilon > 0 \ ;$$

then $v_1 - \dfrac{\varepsilon}{\beta} \leq \limsup\limits_{n\to\infty} t_{1,n}(\beta, \gamma_n)$.

Now, let us prove that:

$$\limsup_{n\to\infty} v_{1,n}(\gamma_n + 2\beta c) \leq v_1(2\beta c) \ .$$

Indeed, under assumptions (5.1), (5.3) and (5.4):

$$\operatorname*{Limsup}_{n\to\infty} M_2(x_n, \varepsilon_n + \varepsilon) \subseteq M_2(x, \varepsilon)$$

for any sequence $(x_n)_n$ converging to x, any $\varepsilon \geq 0$ and any sequence $\varepsilon_n \searrow 0^+$ (only but an adaptation of the proof given in Proposition 4.1 [25]), then, as in Proposition 5.1 in [26] we can prove that:

$$\limsup_{n\to\infty} v_{1,n}(\varepsilon_n + \varepsilon) \leq v_1(\varepsilon)$$

for any ε_n converging to zero and $\varepsilon > 0$. Finally, from $t_{1,n}(\beta_n, \gamma_n) \leq v_{1,n}(\gamma_n + 2\beta_n c)$ and Proposition 5.2 in [26], we get the second result. ■

Remark. Under assumptions (5.1), (5.3) and (5.4) we have also:

$$\operatorname*{Limsup}_{n} M_n(x_n, \beta_n, \gamma_n) \subseteq M_2(x) .$$

Now, let us consider β_k decreasing to zero. We have:

PROPOSITION 15 *Under assumptions of Proposition 14 and*

(3.4) $y \to f_2(x, y)$ *is sequentially lower semicontinuous on* Y *for any* $x \in X$,

(3.5) $y \to f_1(x, y)$ *is sequentially upper semicontinuous on* Y *for any* $x \in X$,

we have:

$$\lim_{k\to\infty} \limsup_{n\to\infty} t_{1,n}(\beta_k, \gamma_n) = v_1$$

for any $\gamma_n \searrow 0^+$ *and* $\beta_k \searrow 0^+$.

Proof: Let $\beta_k \searrow 0^+$ and $\alpha_k = \beta_k^2$.
From inequalities given by (5.7)

$$v_1 \leq \lim_{k\to\infty} \limsup_{n\to\infty} t_{1,n}(\beta_k, \gamma_n) \leq \ldots \leq \lim_{n\to\infty} v_1(2\beta_k c)$$

and we can conclude by using Proposition 5.

■

We also can obtain a best lower bound for $\limsup\limits_{n\to\infty} t_{1,n}(\beta, \gamma_n)$ and a new convergence result for $t_{1,n}(\beta_k, \gamma_n)$.

PROPOSITION 16 *Let* V *be a vectorial topological space and* Y *be a convex compact subset of* V. *For* $\gamma_n \searrow 0^+$, *under assumptions of Proposition 14, (3.4), (3.5) and*

(5.11) $y \to f_2(x, y)$ *is strictly quasi convex on* Y *for any* $x \in X$ *and sequentially lower semicontinuous on* Y

then

$$(5.12) \quad v_1(\varepsilon) - \frac{\varepsilon}{\beta} \le \liminf_{n\to\infty} t_{1,n}(\beta,\gamma_n) \le \dots \le \limsup_{n\to\infty} t_{1,n}(\beta,\gamma_n) \le v_1(2\beta c)$$

for any $\varepsilon > 0$ *and* $\beta > 0$.

Proof: Taking into account the proof of Proposition 14 it is sufficient to apply Poposition 3.8 in [27].

■

PROPOSITION 17 *If the assumptions of Proposition 16 are satisfied, for* $\beta_n \searrow 0^+$, $\gamma_n \searrow 0^+$, *we have:*

$$(5.13) \quad \lim_{k\to\infty}\left(\lim_{n\to\infty} t_1(\beta_n,\gamma_k)\right) = v_1 \ .$$

Proof: Let $\beta_n \searrow 0^+$ and $\alpha_k = \beta_k^2$.
From (5.12)

$$v_1 = \lim_{k\to\infty}(v_1(\alpha_k) - \beta_k) \le \left\{\begin{array}{c}\liminf\limits_{k\to\infty} \\ \\ \limsup\limits_{k\to\infty}\end{array}\right\} \liminf_{n\to\infty} t_{1,n}(\beta_k,\gamma_n) \le$$

$$\dots\left\{\begin{array}{c}\liminf\limits_{k\to\infty} \\ \\ \limsup\limits_{k\to\infty}\end{array}\right\} \limsup_{n\to\infty} t_{1,n}(\beta_k,\gamma_n) \le \lim_{n\to\infty} v_1(2\beta_n c) = v_1$$

and we get the result.

■

COROLLARY 2 *If the assumptions of Proposition 17 are satisfied, for any* $\gamma_n \searrow 0^+$ *there exists an increasing sequence* $k(n)$ *of integers converging to* $+\infty$ *such that*

$$(5.14) \quad \lim_{n\to\infty} t_{1,n}(\beta_{k(n)},\gamma_n) = v_1 \ .$$

Remark. In Proposition 15 and 17, by using Proposition 5, assumptions (3.4) and (3.5) can be substituted by assumptions (3.1) to (3.3).

6. Molodtsov marginal function under perturbations and convergence of solution

Let us determine convergence results for the marginal function $m_{1,n}(\cdot,\beta,\gamma)$ of the problem $Q_n^{\beta,\gamma}$.

Let us recall that for existence of a solution to $Q_n^{\beta,\gamma}$ it can be applied Proposition 4.2 to the functions $f_{1,n}$ and $f_{2,n}$.

A - Vertical perturbations

When X and Y are compact subsets of finite dimensional Euclidean spaces, a first result concerning $m_{1,n}(\cdot, \beta_n, \gamma_n)$ (as $\beta_n \searrow 0^+$, $\gamma_n \searrow 0^+$, $\frac{\gamma_n}{\beta_n} \to 0^+$) can be obtained when the considered perturbations on the lower level problem correspond to the so called "vertical perturbations" of mathematical programming ([18]). More precisely, for $\varepsilon_n \searrow 0^+$ we have:

$$Q_n^{\beta_n,\gamma_n} \begin{cases} \operatorname*{Min}_{x \in X} \inf_{y \in M_n(x,\beta_n,\gamma_n)} f_1(x,y) \\ \text{where } M_n(x,\beta_n,\gamma_n) \text{ is the set of } \gamma_n\text{-solutions to:} \\ R_n^{\beta_n}(x) \begin{cases} \operatorname*{Min}_{\substack{g_i(x,y) \le \varepsilon_n \ \ i=1,q \\ y \in Y}} f_2(x,y) - \beta_n f_1(x,y) \end{cases} \end{cases}$$

associated to the following so called weak bilevel programming problem

$$S \begin{cases} \operatorname*{Min}_{x \in X} \sup_{y \in M_2(x)} f_1(x,y) \\ \text{where } M_2(x) \text{ is the set of solutions to:} \\ R(x) \begin{cases} \operatorname*{Min}_{\substack{g_i(x,y) \le 0 \ \ i=1,q \\ y \in Y}} f_2(x,y) \end{cases} \end{cases}$$

Again, let us denote $t_{1,n}(\beta_n, \gamma_n) = \inf_{x \in X} \inf_{y \in M_n(x,\beta_n,\gamma_n)} f_1(x,y)$.

First, for $\varepsilon \geq 0$, let us give preliminary results connecting the set $T_n(x, \varepsilon)$ of the ε-solutions to the perturbed mathematical programming problem, where $g_{i,n}$ is a general perturbation of g_i

$$\begin{cases} \operatorname*{Min}_{\substack{g_{i,n}(x,y) \le 0 \ \ i=1,q \\ y \in Y}} f_n(x,y) \end{cases}$$

and the set $T(x, \varepsilon)$ of the ε-solutions to the original one

$$\begin{cases} \operatorname*{Min}_{\substack{g_i(x,y) \le 0 \ \ i=1,q \\ y \in Y}} f(x,y) \end{cases}$$

LEMMA 1 *(from Theorem 4.1 in [24]).*
If the following assumptions are satisfied

(6.1) f_n *is continuously convergent to* f, *i.e.* $\lim_{n\to\infty} f_n(x_n, y_n) = f(x,y)$, *for any sequence* $(x_n, y_n)_n$ *converging to* (x,y);

(6.2) $g_{i,n}$ *is continuously convergent to* g_i *for* $i = 1, q$;

(6.3) for any $x \in X$ *there exists* $y \in Y$ *such that* $g_i(x,y) < 0$ *for* $i = 1, q$;

(6.4) for $i = 1, q$, $n \in \mathbf{N}$, $x \in X$ *the function* $y \to g_{i,n}(x,y)$ *is strictly quasi convex and continuous on* Y;

then $\operatorname{Limsup}_n T_n(x_n, 0) \subseteq T(x, 0)$ *for any* $x \in X$ *and any sequence* $(x_n)_n$ *converging to* x.

LEMMA 2 *([24], Theorem 5.3 and Remark 5.1)* .
If assumptions (6.1) to (6.4) and the following are satisfied

(6.6) $\quad y \to f(x,y)$ *is strictly quasi convex on* Y *for any* $x \in X$,

then:

$$T(x,\varepsilon) \subseteq \operatorname{Liminf}_n T_n(x_n, \varepsilon)$$

for any $\varepsilon > 0$, *any* $x \in X$ *and any sequence* $(x_n)_n$ *converging to* x.

For what concerns "vertical perturbations", we first state the two following propositions:

PROPOSITION 18 *Let* $\alpha_n \searrow 0^+$. *If the following assumptions are satisfied:*

(6.5) for any $x \in X$ *there exists* $y \in Y$ *such that* $g_i(x,y) < 0$ *for* $i = 1, q$;

(6.6) f_2 *and* g_i *are continuous on* $X \times Y$ *for* $i = 1, q$;

(6.7) $x \to g_i(x,y)$ *is strictly quasi convex on* X *for any* $y \in Y$ *and* $i = 1, q$;

(6.8) $y \to g_i(x,y)$ *is convex for any* $x \in X$;

(6.9) $y \to f_2(x,y)$ *is convex for any* $x \in X$;

then

i) $\quad \operatorname{Limsup}_n M_{2,n}(x_n) \subseteq M_2(x)$, *where* $M_{2,n}$ *is the set of solutions to*

$$\begin{cases} \displaystyle\operatorname*{Min}_{\substack{g_i(x,y) \le \alpha_n \quad i=1,q \\ y \in Y}} \quad f_2(x,y) \end{cases}$$

and $M_2(x)$ *the set of solutions to*

$$\left\{ \begin{array}{ll} \underset{\substack{g_i(x,y)\leq 0 \quad i=1,q \\ y\in Y}}{\text{Min}} & f_2(x,y) \end{array} \right.$$

for any $x \in X$ and any sequence (x_n) converging to x.

ii) *For any $x \in X$ there exists $\varepsilon_n(x) = r(x)\alpha_n$ such that:*

$$M_2(x) \subseteq \operatorname*{Liminf}_n M_{2,n}(x, \varepsilon_n(x)) \; .$$

Proof:
i) is a consequence of lemma 1.
Prove ii): let $\overline{y} \in M_2(x)$ that is $g_i(x,\overline{y}) \leq 0$ for $i = 1, q$ and $f_2(x,\overline{y}) \leq f_2(x,y)$ for any $y \in Y$ such that $g_i(x,y) \leq 0$ for $i = 1, q$. Then, by using an exact penalty technique as in [41] we have, under assumptions (6.3), (6.8) and (6.9): there exists $r(x) \in \mathbf{R}^+$ such that:

$$f_2(x,\overline{y}) \leq f_2(x,y) + \frac{r(x)}{q} \sum_{i=1}^{q} [g_i(x,y)]^+$$

for any $y \in Y$ with $[g_i(x,y)]^+ = \max[g_i(x,y)), 0]$. This implies:

$$f_2(x,\overline{y}) \leq f_2(x,y) + \frac{r(x)}{q} \left[\sum_{i=1}^{q} (g_i(x,y) - \alpha_n)^+ \right] + r(x)\alpha_n$$

and $f_2(x,\overline{y}) \leq f_2(x,y) + r(x)\alpha_n$ for any $y \in Y$ such that $g_i(x,y) \leq \alpha_n$ for $i = 1, q$.
■

PROPOSITION 19 *If the assumptions of Proposition 18 and the following are satisfied:*

(6.10) $y \to f_1(x,y)$ *is continuous on Y for any $x \in X$,*

we have

i) for any $x \in X$ there exists $\varepsilon_n(x) = r(x)\alpha_n$ such that:

$$\liminf_{n\to\infty} w_{1,n}(x, \varepsilon_n(x)) \geq w_1(x) \; ;$$

ii) for any $x \in X$, for any $\xi_n \searrow 0^+$

$$\limsup_{n\to\infty} w_{1,n}(x, \xi_n) \leq w_1(x) \; .$$

Proof: Let us prove point i): let $x \in X$ and $\varepsilon_n(x)$ as defined in ii) of Proposition 18. For any $\overline{y} \in M_2(x)$ there exists $(\overline{y}_n)_n$ converging to $\overline{y}$ such that:

$$\overline{y} \in M_{2,n}(x, \varepsilon_n(x)) \ .$$

Therefore:

$$w_{1,n}(x, \varepsilon_n(x)) \geq f_1(x, \overline{y}_n)$$

and, from (6.10), we get:

$$\liminf_{n \to \infty} w_{1,n_k}(x, \varepsilon_n(x)) \geq f_1(x, \overline{y})$$

and the result follows.

Prove part ii): let $x \in X$ such that $w_1(x) < +\infty$. We have $w_{1,n}(x, \xi_n) < +\infty$, at least for n sufficiently large. Then, for any $\eta > 0$, there exists a sequence $(\overline{y}_n)_n$ verifying $\overline{y}_n \in M_n(x, \xi_n)$ such that $w_{1,n}(x, \xi_n) \leq f_1(x, \overline{y}_n) + \eta$ for n sufficiently large. Y being compact, there exists a subsequence $(\overline{y}_{n_k})_k$ converging to $\overline{y} \in M_2(x)$ as $k \to +\infty$. Next, by using assumption (6.10), we get:

$$\limsup_{k \to \infty} w_{1,n}(x, \xi_{n_k}) \leq \dots$$
$$\dots \leq \limsup f_1(x, \overline{y}_{n_k}) \leq f_1(x, \overline{y}) + \eta \leq w_1(x) + \eta \ .$$

Now, it is easy to prove that the previous result also holds for the entire sequence $(w_{1,n}(x, \xi_n))_n$, that is to say:

$$\limsup_{n \to \infty} w_{1,n}(x, \xi_n) \leq w_1(x) \ .$$

■

Now, we are able to give a pointwise convergence result for $m_{1,n}(\cdot, \beta_n, \gamma_n)$ to w_1, where $m_{1,n}$ is defined by (5.8) and vertical perturbations are considered in the lower level problem.

PROPOSITION 20 *Let $\beta_n \searrow 0^+$, $\gamma_n \searrow 0^+$ such that $\frac{\gamma_n}{\beta_n} \to 0^+$ and $\alpha_n = \beta_n^2$. Assume that the assumptions of Proposition 19 are satisfied. Then we have:*

$$m_{1,n}(x, \beta_n, \gamma_n) \to w_1(x) \quad \textit{for any} \quad x \in X \ .$$

Proof: From inequalities (5.9), for any $x \in X$,

$$w_1(x, \varepsilon) - \frac{\varepsilon + \gamma_n}{\beta_n} \leq m_{1,n}(x, \beta_n, \gamma_n) \leq \dots$$
$$\dots \leq w_{1,n}(x, \gamma_n + 2\beta_n c) \quad \text{for any} \quad \varepsilon \geq 0 \ .$$

Let $\varepsilon_n(x) = r(x)\beta_n^2$ with $r(x)$ as defined in Proposition 18, then from Proposition 19,

$$\begin{aligned}
&w_1(x) \le \liminf_{n\to\infty} w_{1,n}(x, \varepsilon_n(x)) \le \dots \\
&\le \liminf_{n\to\infty} m_{1,n}(x, \beta_n, \gamma_n) \le \limsup_{n\to\infty} m_{1,n}(x, \beta_n, \gamma_n) \\
&\le \limsup_{n\to\infty} w_{1,n}(x, \gamma_n, 2\beta_n c) \le w_1(x)
\end{aligned}$$

and the result follows.

■

B - General perturbations

Unfortunately pointwise convergence of $m_{1,n}(\cdot, \beta_n, \gamma_n)$ to w_1 is not generally obtained (for any $\beta_n \searrow 0^+$, $\gamma_n \searrow 0^+$ such that $\frac{\gamma_n}{\beta_n} \to 0^+$) for more general perturbations. However we can give a trivial partial result on epiconvergence.

In the following X and Y will be again sequentially compact subsets of topological spaces and the original problem S and the perturbed one $Q_n^{\beta,\gamma}$ are the problems defined respectively in the introduction and the beginning of section 5.

PROPOSITION 21 *Under assumptions (5.1) to (5.6) we have*

(6.11) *for any $x \in X$, there exists a sequence $(\overline{x}_n)_n$ converging to x such that for any $\beta_n \searrow 0^+$ and any $\gamma_n \searrow 0^+$ with $\frac{\gamma_n}{\beta_n} \searrow 0^+$:*

$$\limsup_{n\to\infty} m_{1,n}(\overline{x}_n, \beta_n, \gamma_n) \le w_1(x) .$$

Proof: The proof is straightforward by using:
$\operatorname{Limsup}_n M_{2,n}(x_n, \varepsilon_n) \subseteq M_2(x)$ for any $x \in X$ and any sequence (x_n) converging to x (Remark 4.2 and Proposition 5.2 in [26]) and $M_n(x_n, \beta_n, \gamma_n) \subseteq M_{2,n}(x_n, \gamma_n + 2\beta_n c)$. ■

PROPOSITION 22 *Let $\gamma_n \searrow 0^+$ and Y be a first countable topological space. Under assumptions (5.1) to (5.6) and (3.5) we have:*

i) for any $x \in X$ and any sequence (x_n) converging to x, for any fixed $\beta > 0$:

$$\liminf_{n\to\infty} m_{1,n}(x_n, \beta, \gamma_n) \ge w_1(x) ;$$

ii) for any $x \in X$, any sequence (x_n) converging to x and any sequence (β_k) converging to zero:

$$\liminf_{k\to\infty} (\liminf_{n\to\infty} m_{1,n}(x_n, \beta_k, \gamma_n)) \ge w_1(x) .$$

Proof: Prove i). From (5.9), for any $\varepsilon_n \geq 0$

$$w_{1,n}(x_n, \varepsilon_n) - \frac{\varepsilon_n + \gamma_n}{\beta} \leq m_{1,n}(x_n, \beta, \gamma_n) \ .$$

By choosing $\varepsilon_n^*(x_n, x) \to 0^+$ such that

$$w_1(x) \leq \liminf_{n\to\infty} w_{1,n}(x_n, \varepsilon_n^*) \ ,$$

(Proposition 5.4 in [26]) we get the result.
ii) is obvious.

■

For the solutions we have:

PROPOSITION 23 *Let Y be sequentially compact, $\alpha_n \searrow 0^+$, $\varepsilon_n \searrow 0^+$, $\beta_n \searrow 0^+$, $\gamma_n \searrow 0^+$ such that $\frac{\varepsilon_n}{\beta_n} \to 0^+$ and $\frac{\gamma_n}{\beta_n} \to 0^+$. Under assumptions (5.1) to (5.6) if x_n is an α_n solution to the problem $Q_n^{(\beta_n,\gamma_n)}$, $y_n \in M_{2,n}(x_n, \varepsilon_n)$ and $((x_n, y_n)$ is convergent to (x^*, y^*) then (x^*, y^*) is a lower Stackelberg equilibrium for S.*

Proof: x_n and y_n are such that:

$$\begin{aligned} &m_{1,n}(x_n, \beta_n, \gamma_n) \leq t_{1,n}(\beta_n, \gamma_n) + \alpha_n \\ &f_{1,n}(x_n, y_n) \leq w_{1,n}(x_n, \varepsilon_n) = \sup_{y \in M_{2,n}(x_n,\varepsilon_n)} f_{1,n}(x_n, y) \ ; \end{aligned}$$

but, from (5.9):

$$w_{1,n}(x_n, \varepsilon_n) \leq m_{1,n}(x_n, \beta_n, \gamma_n) + \frac{\varepsilon_n + \gamma_n}{\beta_n} \ ;$$

then

$$f_{1,n}(x_n, y_n) \leq t_{1,n}(\beta_n, \gamma_n) + \alpha_n + \frac{\varepsilon_n + \gamma_n}{\beta_n}$$

and, by using (5.10):

$$\limsup_{n\to\infty} f_{1,n}(x_n, y_n) \leq \lim v_{1,n}(\gamma_n + 2\beta_n c) \ .$$

So, from Proposition 5.2 in [26] we get the result.

■

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